Math 131-01 - Dr. Miller - Final Exam - 12/10/01

This exam is worth *150 points*. Show clear work where appropriate.

1. Consider this theorem: If a and b are both multiples of 4, then their average $\frac{a+b}{2}$ is even.
   (a) *[3 pts]* If you wanted to prove this theorem *directly*, what would you assume about a, b, or $\frac{a+b}{2}$, and what would you try to show?
   (b) *[3 pts]* If you wanted to prove the theorem by *contrapositive*, what would you assume and what would you try to show?

2. *[6 pts - 2 each]* Identify the hypothesis of each statement below.
   (a) If it rains tomorrow, then I can't go hiking.
   (b) Kathy only goes hiking if it's a weekend.
   (c) My friends like hiking.

3. (a) *[4 pts]* Use a truth table to show that "$p \Rightarrow q$" is logically equivalent to "$\overline{p} \vee q$," stating in a brief sentence how your table confirms this claim.
   (b) *[2 pts]* Simplify this negation: $\overline{p \Rightarrow q}$.

4. (a) *[4 pts]* Translate into entirely symbolic form: "For every real number x there exists an integer n for which x is between n and $n+1$, inclusive."
   (b) *[4 pts]* Symbolically negate "$(\forall a \in \mathbf{R}, a^2 \geq 0) \wedge (\exists b \in \mathbf{Z}, 2b = 1)$"; simplify your answer.

5. *[14 pts - 2 each]* Let $U = \{1,2,3,4,5,6,7,8,9,10\}$ be a universal set with subsets $A = \{2,3,5,7\}$, $B = \{1,2,3\}$ and $C = \{2,4,6,8,10\}$. List the elements of the following sets.
   (a) $A \cup B$
   (b) $B \cap C$
   (c) $B \setminus A$
   (d) $\overline{B}$
   (e) $B \times \{3,4\}$
   (f) $\mathcal{P}(B)$, the power set of B
   (g) Compute $\mathcal{C}_{A\setminus(B\cap C)}(2)$ where $\mathcal{C}$ represents the characteristic function.

6. *[6 pts - 2 each]* Classify the following as true or false, justifying your answers.
   (a) $3 \subseteq \{1,2,3\}$.
   (b) $\emptyset \subseteq \{1,2,3\}$.
   (c) $0 \in \emptyset$.

7. *[8 pts]* Use mathematical induction to prove that $n(n^2+5)$ is a multiple of 6 for all natural numbers n.

continued

8. *[8 pts - 2 each]* Consider the relation on $\{-1, 1, 2\}$ given by xRy if $|x - y| \leq 2$.
   (a) Draw the digraph for this relation.
   (b) Is the relation reflexive? Justify your answer.
   (c) Is the relation symmetric? Justify your answer.
   (d) Is the relation transitive? Justify your answer.

9. (a) *[3 pts]* Make up a relation R from $\{1, 2, 3\}$ to $\{a, b, c\}$ that is **not** a function, explaining how you know.
   (b) *[3 pts]* Make up a **function** f from $\{1, 2, 3\}$ to $\{a, b, c\}$ that is **not** one-to-one, explaining how you know.
   (c) *[3 pts]* Is the function f that you just created onto $\{a, b, c\}$? Explain.
   (d) *[3 pts]* List the ordered pairs in the composition $f \circ g$ where f is your function above and g is the altered ceiling function $g(x) = \lceil \frac{x}{2} \rceil$ on the set $\{1, 2, 3\}$.

10. *[6 pts - 2 each]* Consider the sequence defined by $s_n = n(n - 1)$.
   (a) List the values of s_2, s_3, and s_4.
   (b) Evaluate $\prod_{i=2}^{4} s_i$.
   (c) Evaluate $\sum_{i=1}^{3} s_{(i+2)}$.

11. *[5 pts]* Show that for the Fibonacci sequence, $f_n^2 - 2f_{n-1}f_{n-2} = f_{n-1}^2 + f_{n-2}^2$.

12. (a) *[3 pts]* Use the Euclidean Algorithm to compute the greatest common divisor of 731 and 34. Clearly indicate your final answer.
   (b) *[2 pts]* Evaluate 250 mod 7.
   (c) *[3 pts]* Evaluate 2^{101} mod 30; show clear work.
   (d) *[3 pts]* Evaluate $\binom{50}{47}$ by hand; show work.

13. I have 3 dozen doughnuts: 12 chocolate, 12 glazed, and 12 jelly-filled.
   (a) *[2 pts]* In how many ways can I pick 6 doughnuts to serve for breakfast?
   (b) *[2 pts]* My family hates jelly-filled doughnuts. If I have to avoid those, in how many ways can I now select 6 doughnuts to serve for breakfast?
   (c) *[4 pts]* In how many ways can I pick 3 doughnuts each of two types? (Jelly-filled are okay again.)
   (d) *[4 pts]* In how many ways can I pick 3 of one type, 2 of another, and 1 of the third?

14. (a) *[4 pts]* Find the coefficient of $a^6b^3c^2$ in $(2a - b + c)^3(2a - b)^8$. Answer in symbolic form.
   (b) *[4 pts]* Use the Binomial Theorem to prove that $2^n = \sum_{k=0}^{n} \binom{n}{k}$.

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15. (a) *[2 pts]* Draw a graph that has 5 vertices: 1 of degree 0, 2 of degree 3, and 2 of degree 4.
    (b) *[2 pts]* Does your graph contain any isolated vertices? Justify your answer.
    (c) *[2 pts]* Does your graph contain any parallel edges? Justify.
    (d) *[2 pts]* Is your graph complete? Justify.

16. *[12 pts - 4 each]* Consider the following graph:

    (a) Does it contain an Euler cycle? Justify your answer.
    (b) Does it contain a Hamiltonian cycle? Justify your answer.
    (c) Sketch a spanning tree for this graph.

17. *[14 pts - 2 each]* Consider the following tree:

    (a) Redraw it as a rooted tree with vertex a as the root.
    (b) What is the height of your rooted tree?
    (c) List the parent(s), if any, of vertex b.
    (d) List the child(ren), if any, of vertex b.
    (e) List the sibling(s), if any, of vertex b.
    (f) List the ancestor(s), if any, of vertex i.
    (g) List the terminal vertices of the tree.