Key $\frac{26}{26}$

Math 131 - Dr. Miller - Exam #2 - 11/2/11

This exam is worth 100 points.

1. *[18 pts - 6 each]* Evaluate the following, giving each final answer as a single integer or fraction:

(a) $\sum_{i=3}^{6} \frac{i!}{2i}$ $= \frac{3!}{6} + \frac{4!}{8} + \frac{5!}{10} + \frac{6!}{12}$

$= 1 + 3 + 12 + 60$

$= \boxed{76}$

(b) $\left(1-\frac{1}{3}\right)\cdot\left(1-\frac{1}{4}\right)\cdot\ldots\left(1-\frac{1}{n}\right)$ for $n=6$

$\frac{2}{\cancel{3}} \cdot \frac{\cancel{3}}{\cancel{4}} \cdot \frac{\cancel{4}}{\cancel{5}} \cdot \frac{\cancel{5}}{6} = \boxed{\frac{2}{6} \text{ or } \frac{1}{3}}$

(-4) $\frac{5}{6}$

(c) $2^2 + 2^3 + \ldots + 2^{n-1}$ for $n = 3$.

$2^2 = \boxed{4}$

2. *[8 pts]* Rewrite the expression below using sigma or pi notation:

$$\frac{16}{2^5} \cdot \frac{25}{2^6} \cdot \frac{36}{2^7} \cdot \ldots \cdot \frac{10000}{2^{101}}$$

(-1) mixed variables
(-2) too far out.

$$\boxed{\prod_{n=5}^{101} \frac{(n-1)^2}{2^n} \quad \text{or} \quad \prod_{n=4}^{100} \frac{n^2}{2^{n+1}}}$$

$\frac{20}{20}$

3. *[10 pts]* Find an explicit, position-based formula for the sequence a_n below, showing work as needed. Specify the domain for your formula as well.

$$\frac{5}{2}, \frac{-10}{3}, \frac{15}{4}, \frac{-20}{5}, \frac{25}{6}, \ldots$$

For $n \in \mathbb{Z}, n \geq 1$, $\quad a_n = (-1)^{n+1} \cdot \frac{5n}{n+1}$

or

For $n \in \mathbb{Z}, n \geq 2$, $\quad a_n = (-1)^n \cdot \frac{5(n-1)}{n}$

(-2) n+1 and n-1
(-1) omitted integers
(-1) mixed variables
(-3) n+5
(-3) 5^n
(-2) bad domain cut-off or no domain.
(-3) no -1 to power.
(-1) -1^n
(-3) no

4. *[10 pts]* A breeding pair of rabbits is born at the end of the year. All rabbits are fertile during their second and third months of life, giving birth to 3 more pairs of rabbits at the end of those months. During the fourth month of life, rabbits become infertile. If no rabbits ever die, how many are alive at the end of six months? Show clear work.

(-2) 66

(-2) bad interpret. error.

Month	Newborn (end)	Juvenile (end)	Fertile$_1$ (during)	Fertile$_2$ (during)	Infertile (during)	Total (end)
0	1					
1	0	1				
2	3	0	1			
3	3	3	0	1		
4	9	3	3	0	1	
5	18	9	3	3	1	
6	36	18	9	3	4	$\boxed{70}$

70 pairs are alive at the end of 6 months.

$\frac{14}{14}$

5. For both parts of this problem, let S_n be defined as follows:
   Recurrence relation: $S_n = S_{n-2} - S_{n-3}$ for all integers $n \geq 3$
   Initial conditions: $S_0 = 1$, $S_1 = 1$, $S_2 = 2$

① $S_4 - S_3$

(a) *[4 pts]* Find the value of S_6. Show work.

$$S_3 = S_1 - S_0 = 1 - 1 = 0$$
$$S_4 = S_2 - S_1 = 2 - 1 = 1$$
$$S_5 = S_3 - S_2 = 0 - 2 = -2$$
$$S_6 = S_4 - S_3 = 1 - 0 = 1$$
$$\boxed{S_6 = 1}$$

(-7) individual cases.
(-5) Fibonacci

(b) *[10 pts]* Confirm that the sequence satisfies the equality $S_n + S_{n-1} = S_{n-2} - S_{n-4}$ for all integers $n \geq 4$. Show clear work. (Do not use induction.)

$$S_n + S_{n-1} \stackrel{?}{=} S_{n-2} - S_{n-4}$$
$$S_{n-2} - S_{n-3} + S_{n-1} \stackrel{?}{=} \quad ''$$
$$S_{n-2} - \cancel{S_{n-3}} + \cancel{S_{n-3}} - S_{n-4} \stackrel{?}{=} \quad ''$$
$$S_{n-2} - S_{n-4} = S_{n-2} - S_{n-4}$$

6. *[40 points]* <u>CHOOSE</u> TWO of the following three induction problems to be graded for 20 points each; work them on the blank pages provided. The third problem will NOT be graded. (No, NO bonus opportunity, and no, I WON'T just count your "best" two out of three.)

Prove using mathematical induction: $\prod_{i=1}^{n} \frac{i}{i+1} = \frac{1}{n+1}$ for all integers $n \geq 1$.

Prove using mathematical induction: $2^{3n} - 1$ is divisible by 7 for all integers $n \geq 0$.

Prove using mathematical induction: $\sum_{i=1}^{n} i(i!) < (n+1)! + 1$ for all integers $n \geq 2$.

$\frac{20}{20}$

#6a. Let $P(n)$ be the statement $\prod_{i=1}^{n} \frac{i}{i+1} = \frac{1}{n+1}$.

(-2) wrong formula (3)

(-2) failed, ≠

(3) $P(1)$ is true because $\prod_{i=1}^{1} \frac{i}{i+1} = \frac{1}{2} = \frac{1}{1+1}$

(4) Assume $P(k)$ is true where k is an integer and $k \geq 1$.

(-1) n showing

That is,

$$\prod_{i=1}^{k} \frac{i}{i+1} = \frac{1}{k+1}.$$

(-2)

(8) We'll show that $P(k+1)$ is true; that is, that

$$\prod_{i=1}^{k+1} \frac{i}{i+1} \overset{?}{=} \frac{1}{k+2}.$$

(-2) n showing

(-2) $\frac{i+1}{i+2}$, not $\frac{k+1}{k+2}$

(-3) wrong formula

$$\prod_{i=1}^{k} \frac{i}{i+1} \cdot \frac{k+1}{k+2} \overset{?}{=} \text{"}$$

$$\frac{1}{\cancel{k+1}} \cdot \frac{\cancel{k+1}}{k+2} \overset{?}{=} \text{"}$$

$$\frac{1}{k+2} = \frac{1}{k+2}$$

(2) By PMI, $P(n)$ is true for all integers $n \geq 1$.

#6b. (2) Let $P(n)$ be the statement $2^{3n}-1$ is divisible by 7.

(-1) "$\frac{0}{7}=0$ so true"
(-3) "$0=0$" "$7=7$"

(3) $P(0)$ is true because $2^0-1=0$ is a multiple of 7; it's of the form $0=7\cdot 0$, the product of 7 and the integer 0.

(-2) no assume

(3) Assume $P(k)$ is true. That is, $2^{3k}-1$ is divisible by 7, so that $2^{3k}-1=7\cdot c$ for some integer c.

(-2) no $c \in \mathbb{Z}$

(10) Will show that $P(k+1)$ is true: that $2^{3(k+1)}-1$ is divisible by 7.

$$\begin{aligned} 2^{3(k+1)}-1 &= 2^{3k+3}-1 \\ &= 8\cdot 2^{3k}-1 \\ &= 8(2^{3k}-1)+8-1 \\ &= 8(7c)+7 \\ &= 7(8c+1) \end{aligned}$$

Because $8c+1 \in \mathbb{Z}$, $2^{3(k+1)}-1$ is divisible by 7.

(-2) no $8c+1 \in \mathbb{Z}$
(-1) wrong reference to $\mathbb{Z}$
(-3) -2 no. $+8$
(-2) $2^{3k+3}=2\cdot 2^{3k}$

(2) By PMI, $P(n)$ is true for all integers $n \geq 0$.

#6c. (2) Let $P(n)$ be the statement $\sum_{i=1}^{n} i(i!) < (n+1)!+1$.

(-2) bad value (3) We know $P(2)$ is true because

$$\sum_{i=1}^{2} i(i!) = 1\cdot 1! + 2\cdot 2! = 5 < (2+1)!+1 = 6+1 = 7.$$

(3) Assume $P(k)$ is true. That is, $\sum_{i=1}^{k} i(i!) < (k+1)!+1$.

(10) We'll show that $P(k+1)$ is true: that $\sum_{i=1}^{k+1} i(i!) < (k+2)!+1$.

$$\sum_{i=1}^{k+1} i(i!) = \sum_{i=1}^{k} i(i!) + (k+1)\cdot(k+1)!$$

(-6) lost $(k+1)(k+1)!$.

(-6) bad algebra.

$$\begin{aligned} &< (k+1)!+1 + (k+1)(k+1)! \\ &= (k+1)!\left[1+k+1\right] + 1 \\ &= (k+1)!\cdot(k+2) + 1 \\ &= (k+2)!+1. \end{aligned}$$

(2) By PMI, $P(n)$ is true for all integers $n \geq 2$.