Math 235 - Dr. Miller - HW #5, Fall 2024 - Due Friday, Oct. 4, 2024, by 4pm

1. (a) There's a rare logical equivalence that allows us to exchange a negation-filled biconditional statement for a less entangled one. Note that this equivalence is NOT a rule for NEGATING a biconditional: the negation of a conditional should NEVER still be conditional! Create a truth table to confirm that

$$p \leftrightarrow q \equiv (\sim p) \leftrightarrow (\sim q)$$

   (b) Now use the equivalence above to rewrite this Proposition: *Let $x, y \in \mathbf{Z}$. Then x and y are not equal if and only if the square of their sum does not equal 4 times their product.*

   (c) Finally, write a chain-style proof of your "nicer" version of the Proposition.

2. Rigorously prove the following Propositions:

   (a) For every integer n, there is an integer M for which $M > 3n$.

   (b) Let $S = \{4k + 1 \mid k \in \mathbf{Z}\}$ and $T = \{4k - 1 \mid k \in \mathbf{Z}\}$. For all $m \in S$, $m + 2 \in T$.

   It's okay to condense universal hypotheses like this one to simply: "Let S and T be as defined."

   (c) Let $b, c \in \mathbf{Z}$. Every common divisor of $b - 1$ and $c - 1$ is also a divisor of $bc - 1$.

   (d) $\mathbf{Q}$ is closed under NON-ZERO division. (Hint: be careful about the non-zero features in your proof.)

   (e) Let $T = \{x \in \mathbf{Z} \mid x \text{ has remainder 4 or 6 on division by } 10\}$. Then T is closed under multiplication. (Reminder: "Let T be as defined.")

3. Write a rigorous, constructive proof of each statement below:

   (a) There exists a prime number x for which $x + 2$, $x + 3$, $x + 4$, and $x + 5$ are composite.
   (b) There exists a composite number y for which $y + 2$ and $y + 4$ are prime.
   (c) There exist integers a, b and c where $a | b + c$ but a divides neither b nor c individually.
   (d) There exists a quadratic equation that has no real solutions.
   (e) The set T of positive factors of 100! is not a subset of $S = \{1, 2, 3, \ldots, 100\}$.