You have already turned in Part A, which was to prove Problems #5g and #7k from the problem set on page 37 of our textbook. Please write each proof below on a SEPARATE sheet of paper, with your name on the top of each page. This will allow me to grade and return problems separately, without the need to wait until all are graded to give you any feedback.

Prove the following statements directly, using correct proof style. Use cases or “wlog” where appropriate. You may assume that all variables represent integers. Use ONLY the definitions that we have given in class.

1. If \( a \mid b \) and \( a \mid (b + c) \), then \( a \mid 3c \).

   \[ \text{Proof. Assume that } a \mid b \text{ and } a \mid (b + c). \] (NTS: \( a \mid 3c \).) Then there exist integers \( x \) and \( y \) for which \( ax = b \) and \( ay = b + c \). By algebra, we have \( ay - ax = b + c - b = c \), so \( 3ay - 3ax = 3c \). Factoring yields \( a(3y - 3x) = 3c \), where \( 3y - 3x \in \mathbb{Z} \) because it is a product and difference of integers. Thus, \( a \mid 3c \). Therefore, if \( a \mid b \) and \( a \mid (b + c) \), then \( a \mid 3c \).

2. If \( x \) has remainder 5 on division by 10, then so does \( x^2 \). (This proves that if a number has a ones digit of 5, then so will its square.)

   \[ \text{Proof. Assume } x \text{ has remainder 5 on division by 10.} \] (NTS: \( x^2 \) has remainder 5 on division by 10.) Then we can write \( x = 10q + 5 \) for some integer \( q \). By algebra, \( x^2 = 100q^2 + 100q + 25 = 100q^2 + 100q + 20 + 5 = 10(10q^2 + 10q + 2) + 5 \). Now \( 10q^2 + 10q + 2 \in \mathbb{Z} \) because it is a sum and product of integers. Thus, \( x^2 \) has remainder 5 on division by 10. Therefore, if \( x \) has remainder 5 on division by 10, then so does \( x^2 \).

3. If \( n \) is a natural number, then \( n^2 + n + 3 \) is odd.

   \[ \text{This one comes from the textbook, so I won’t provide a printed solution. See me if you have questions.} \]

4. If \( x \) and \( y \) have different parity, then \( x + y \) is odd.

   \[ \text{Proof. Suppose that } x \text{ and } y \text{ have different parity, meaning that one of them is even and the other odd.} \] (NTS: \( x + y \) is odd.) Without loss of generality, let \( x \) be the even number and \( y \) odd. Then there exist integers \( m \) and \( n \) for which \( x = 2m \) and \( y = 2n + 1 \). By algebra, \( x + y = 2m + 2n + 1 = 2(m + n) + 1 \), where \( m + n \) is an integer because it is a sum of integers. Thus \( x + y \) is odd. Therefore, if \( x \) and \( y \) have different parity, then \( x + y \) is odd.
5. Let \( x, y, \) and \( z \) be integers. If at least one of them is divisible by 7, then \( xyz \) is divisible by 7.

Proof. Let \( x, y, \) and \( z \) be integers, and assume that at least one of them is divisible by 7. (NTS: \( 7 \mid xyz \).) [You may expect us to tackle cases next based on whether exactly one, exactly two, or all of the three are divisible by 7, however...] Without loss of generality, let \( x \) be divisible by 7. Then there exists an integer \( m \) for which \( x = 7m \), and \( xyz = 7myz \) which is divisible by 7 since the product \( myz \) of integers is an integer. [Notice that it did not actually matter whether \( y \) and \( z \) were divisible or not.] Therefore, if at least one of \( x, y, \) or \( z \) is divisible by 7, then \( xyz \) is divisible by 7.

6. Let \( x, y, \) and \( z \) be integers. If exactly two of them have the same parity, then the third one has the same parity as \( x + y + z \).

Proof. Let \( x, y, \) and \( z \) be integers, and assume that exactly two of them have the same parity. (NTS: the third one has the same parity as \( x + y + z \).)

Case 1: Suppose two of them are even and the other one is odd. (NTS: \( x + y + z \) is odd.) Without loss of generality, let \( x \) and \( y \) be even and \( z \) odd. Then there exist integers \( a, b, \) and \( c \) for which \( x = 2a, y = 2b, \) and \( z = 2c+1 \). By algebra, \( x + y + z = 2a + 2b + 2c + 1 = 2(a + b + c) + 1 \), which is odd because \( a + b + c \in \mathbb{Z} \) since it is a sum of integers.

Case 2: Suppose two of them are odd and the other one is even. (NTS: \( x + y + z \) is even.) Without loss of generality, let \( x \) and \( y \) be odd and \( z \) even. Then there exist integers \( a, b, \) and \( c \) for which \( x = 2a + 1, y = 2b + 1, \) and \( z = 2c \). By algebra, \( x + y + z = 2a + 1 + 2b + 1 + 2c = 2(a + b + c + 1) \), which is even because \( a + b + c + 1 \in \mathbb{Z} \) since it is a sum of integers.

Therefore, if exactly two of \( x, y, \) and \( z \) have the same parity, then the third one has the same parity as \( x + y + z \).