1. (a) Prove that $T = \{8n - 7 \mid n \in \mathbb{Z}\}$ is not a subset of $3\mathbb{Z}$.

Proof. (NTS: there exists $x \in T$ where $x \not\in 3\mathbb{Z}$.) Consider $17 = 8(3) - 7$ (where $3 \in \mathbb{Z}$). $17 \in T$ (because it has the form $8$ times an integer minus $7$), but $17 \not\in 3\mathbb{Z}$. Thus, there exists $x \in T$ where $x \not\in 3\mathbb{Z}$. Therefore, $T \not\subseteq 3\mathbb{Z}$.

(b) Prove that $T$ is a subset of the set $S$ of odd integers.

Proof. Let $x \in T$. (NTS: $x \in S$.) Then $x = 8n - 7 = 2(4n - 4) + 1$. Now $4n - 4 \in \mathbb{Z}$ because it is a product and difference of integers, so $x$ is odd ($= 2k + 1$ for some $k \in \mathbb{Z}$). Thus, $x \in S$. Therefore, $T \subseteq S$.

2. Let $M = \{10a + 15b + 18c \mid a, b, c \in \mathbb{Z}\}$. Prove that $M = \mathbb{Z}$.

Proof. \subseteq: Let $x \in M$. Then $x = 10a + 15b + 18c$ for some integers $a, b,$ and $c$. Then $x \in \mathbb{Z}$ because it is a product and sum of integers. Thus, $M \subseteq \mathbb{Z}$.

\supseteq: Let $x \in \mathbb{Z}$. Then $x = 1x = (2 \cdot 10 + 15 - 2 \cdot 18)x = 10(2x) + 15x + 18(-2x)$. Now $2x, x,$ and $-2x$ are integers because they are the product of integers (actually, $x \in \mathbb{Z}$ by hypothesis), so $x \in M$ (because it’s of the right form). Thus, $\mathbb{Z} \subseteq M$.

Therefore, $M = \mathbb{Z}$.

3. Let $A, B,$ and $C$ be sets with universal set $U$. Prove:

(a) $(A \setminus B) \setminus C = A \setminus (B \cup C)$

Proof. Let $A, B,$ and $C$ be sets with universal set $U$.

$x \in (A \setminus B) \setminus C$ iff $x \in A \setminus B$ but $x \not\in C$

iff $x \in A$ but $x \not\in B$ and $x \not\in C$ (**)

iff $x \in A$ and it’s not true that $x \in B$ or $x \in C$ (**)

iff $x \in A$ and it’s not true that $x \in B \cup C$ (**)

iff $x \in A$ and $x \not\in B \cup C$

iff $x \in A \setminus (B \cup C)$.

Instead of the three lines marked (**), I allowed a variation on de Morgan’s Law even though we had not proved it: that $x \not\in B$ and $x \not\in C$ iff $x \not\in B \cup C$. Be sure you recognize whether you short-cutted that fact, and also that the variation is NOT word-for-word de Morgan’s Law (which was about BELONGING to complements, not NOT BELONGING to sets). It’s something we should have proved before we were allowed to use it.

In every correct formal proof, each statement we write needs a reason based on definitions, axioms, or previously proved results, a reason we can actually NAME (like “by algebra” or “by de Morgan”). If you find that justifying a step would require
some narrative “explanation” instead, you are probably doing something wrong.

(b) \((A \cap C) \times B = (A \times B) \cap (C \times B)\)

\textbf{Proof.} Let \(A\), \(B\), and \(C\) be sets with universal set \(U\).
\[
(x, y) \in (A \cap C) \times B \iff x \in A \cap C \text{ and } y \in B
\]
\[
\text{iff } \begin{cases} \text{iff } x \in A \text{ and } x \in C \text{ and } y \in B \\ (x, y) \in A \times B \text{ and } (x, y) \in C \times B \\ (x, y) \in (A \times B) \cap (C \times B) \end{cases}
\]
\[(c) \ A^c \times B^c \subseteq (A \times B)^c, \text{ but that these sets are not equal.} \]

\textbf{Proof.} (We cannot use a chain-style proof here because this is not an identity.)

\(\subseteq:\) Let \(A\), \(B\), and \(C\) be sets with universal set \(U\), and let \(x, y) \in A^c \times B^c\). \((\text{NTS: } (x, y) \in (A \times B)^c.)\) Then \(x \in A^c\) and \(y \in B^c\), so \(x \notin A\) and \(y \notin B\). By definition of Cartesian product, \((x, y) \notin A \times B\), so \((x, y) \in (A \times B)^c\), whence \(A^c \times B^c \subseteq (A \times B)^c\).

\(\supseteq:\) \((\text{NTS: there exists } x \in (A \times B)^c \text{ with } x \notin A^c \times B^c.)\) Consider \(A = \{0\}\) and \(B = \{1\}\) with \(U = \{0, 1\}\), and also consider \(x = (1, 1)\) (it takes this much to set up our candidate). Now \((1, 1) \notin (A \times B)\) because it does not have a first coordinate from \(A\) and a second from \(B\); thus \((1, 1) \in (A \times B)^c\). Also, \((1, 1) \notin A^c \times B^c:\) although its first coordinate \(1 \in A^c\), its second coordinate \(1 \notin B^c\). Thus, there exists \(x \in (A \times B)^c \text{ where } x \notin A^c \times B^c\), whence \((A \times B)^c \notin A^c \times B^c\) and so \((A \times B)^c \neq A^c \times B^c\). Therefore, \(A^c \times B^c \subseteq (A \times B)^c\), but these sets are not equal.

4. Let \(A\) and \(B\) be sets. Prove that \([\mathcal{P}(A) \cup \mathcal{P}(B)] \subseteq \mathcal{P}(A \cup B)\) but that these sets are not equal.

\textbf{Proof.} Let \(A\) and \(B\) be sets, and suppose \(X \in \mathcal{P}(A) \cup \mathcal{P}(B)\). \((\text{NTS: } X \in \mathcal{P}(A \cup B).)\) By definition of union, \(X \in \mathcal{P}(A)\) or \(X \in \mathcal{P}(B)\), whence \(X \subseteq A\) or \(X \subseteq B\) by definition of power set. \((\text{NTS: } X \subseteq A \cup B.)\) (Be careful using the definition of union: it has nothing to do with subsets, only elements.) Let \(x \in X\).

Case 1: By definition, \(X \subseteq A\) means \(x \in A\) also, and \(x \in A \cup B\) by definition of union.

Case 2: Similarly, \(X \subseteq B\) means \(x \in B\) so that \(x \in A \cup B\).

Thus, \(X \subseteq A \cup B\) and so \(X \in \mathcal{P}(A \cup B)\).

Hence \(\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)\).

\(\supseteq:\) \((\text{NTS: there exists } X \in \mathcal{P}(A \cup B) \text{ where } X \notin \mathcal{P}(A) \cup \mathcal{P}(B).)\)

Consider \(A = \{0\}\) and \(B = \{1\}\) and consider \(X = \{0, 1\}\). Now \(A \cup B = \{0, 1\}\), so certainly \(X \in \mathcal{P}(A \cup B)\) because \(X = A \cup B\). However, \(X \notin \mathcal{P}(A) \cup \mathcal{P}(B)\) for we can list that set and see: \(\mathcal{P}(A) = \{ \emptyset, \{0\} \}, \mathcal{P}(B) = \{ \emptyset, \{1\} \}\), making \(\mathcal{P}(A) \cup \mathcal{P}(B) = \{ \emptyset, \{0\}, \{1\}\}\), and obviously \(X\) does not appear in this set. Thus, there exists \(X \in \mathcal{P}(A \cup B)\) where \(X \notin \mathcal{P}(A) \cup \mathcal{P}(B)\), whence \(\mathcal{P}(A \cup B) \notin \mathcal{P}(A) \cup \mathcal{P}(B)\), so these sets are not equal.

Therefore, \([\mathcal{P}(A) \cup \mathcal{P}(B)] \subseteq \mathcal{P}(A \cup B)\) but these sets are not equal.