Math 235 - Dr. Miller - An Example of “Unnecessary” Proof by Contradiction

Because proof by contradiction seems easy to set up, generations of mathematics students have overused it - crafting a proof by contradiction in a situation where another proof style can be used. This isn’t usually such a terrible thing, but if a proof by contradiction actually has another type of proof EMBEDDED in it, that shows weakness in the writer’s understanding of logic.

As you progress through this course, you should begin to develop not only the abilities to understand and write proofs, but to critique their efficiency. One logical style issue that may begin to show is stating extra hypotheses that you never use. Whenever you see such things in a proof, you should delete them: they are inefficient. Consider this example:

**Theorem:** If \( m^2 \) is even, then \( m \) is even.

**Proof (by contradiction).** Assume that \( m^2 \) is even but \( m \) is odd (this is the correct negation of our statement). Then \( m = 2k + 1 \) for some integer \( k \), whence \( m^2 = 2(2k^2 + 2k) + 1 \), which is odd because \( 2k^2 + 2k \in \mathbb{Z} \). But this contradicts the assumption that \( m^2 \) is even. Therefore by contradiction, if \( m^2 \) is even, then \( m \) is even.

It reads like a perfectly acceptable proof in good style: we assumed \( p \land \sim q \), which is the negation of \( p \Rightarrow q \), and got a contradiction. But look more carefully at the nature of that contradiction: it is just \( \sim p \), when we had assumed \( p \land \sim q \) to be true. It’s the “\( \land \sim q \)” part that’s crucial here: if you assume \( \sim q \) and end up with \( \sim p \), what sort of proof have you really created? It’s a proof by CTP, not contradiction! That whole “assume \( m^2 \) is even” bit could be thrown out and we’d still have a correct proof.

What happened was this: we were proving an implication and **we assumed our original hypothesis \( p \) was true but never USED that information except at the point of contradiction. When that happens, you may have written a proof by ctp, so re-examine and possibly re-write the proof. Useless, extra contradictions are silly and logically unnecessary.**

Try to critique your proofs as we go through the semester to see where you use each hypothesis, and whether you really needed it.