Math 235 - Dr. Miller - Exam #2 - 3/30/12 - **SOLUTIONS**

1. *[10 pts]* Prove carefully and rigorously via an appropriate definition: Let x be a real number. If x is rational, then $x + \frac{1}{2}$ is rational. (You must use a definition; you may NOT simply invoke the homework result about sums of rational numbers.)

Suppose that x is rational. Then $x = \frac{p}{q}$ for some integers p and q with $q \neq 0$. By arithmetic, we know that $x + \frac{1}{2} = \frac{2p+q}{2q}$ which is rational because $2p + q$ and $2q$ are sums and products of integers and therefore also integers, and $2q \neq 0$ because $q \neq 0$.

2. *[10 pts]* Prove carefully and rigorously via an appropriate definition: Let x and y be integers. If xy is odd, then x and y are both odd.

Let x and y be integers and suppose it is not the case that they are both odd. Then at least one of them is even; without loss of generality, let x be even. Then $x = 2n$ for some integer n, and $xy = 2ny = 2(ny)$. Because ny is a product of integers, it is also an integer, so xy is even. By contrapositive, therefore, if xy is odd, then x and y are both odd.

3. *[18 pts - 6 each]* Disprove each statement below, applying appropriate definitions.

(a) If a and b are real numbers with $a^2 < b^2$, then $a < b$.

Consider $a = -3$ and $b = -4$. It is true that $(-3)^2 = 9 < (-4)^2 = 16$ because the positive number 7 satisfies $9 + 7 = 16$. However, $-3 \not< -4$ because there is no positive real number x for which $-3 + x = 4$. (The only solution is $x = -1$, which isn't positive.)

(b) Let $M = \{x \mid x = 10m + 35n + 4p$ for some integers m, n, and $p\}$ and let $K = \{y \mid y = 5k$ for some integer $k\}$. Then $M \subseteq K$.

Consider 4, which is an element of M because $4 = 10(0) + 35(0) + 4(1)$ where 0 and 1 are integers. The number 4 is not an element of K because there is no integer k for which $4 = 5k$. (The only option is $k = 5/4$, which isn't an integer.)

(c) The product of any two distinct irrational numbers is also irrational.

We have already proved that $\sqrt{2}$ is irrational. By similar reasoning $-\sqrt{2}$ is also irrational. Yet their product, -2, is rational, because it can be written as $-2/1$, a fraction of integers with nonzero denominator.

4. *[8 pts]* Let $\Lambda = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\} = \{\frac{1}{n} \mid n \in \mathbf{N}\}$ and define $B_\alpha = [-1, 1-\alpha]$ for each $\alpha \in \Lambda$. Prove that $\cap_\Lambda B_\alpha \subseteq [-1, 0]$.

Let $x \notin [-1,0]$. This interval equals B_1, so $x \notin B_1$. Thus, it is not true that $x \in B_\alpha$ for all $\alpha \in \Lambda$ and therefore $x \notin \cap_\lambda B_\alpha$ by definition of the arbitrary intersection. By contrapositive, $x \in \cap_\lambda B_\alpha$ implies that $x \in [-1, 0]$.

5. *[12 pts - 4 or 1 each]* Let $\Lambda = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$ and define $B_\alpha = [-1+\alpha, 1+\alpha)$ for each $\alpha \in \Lambda$.

   (a) Find $\cap_\Lambda B_\alpha$.

   $$\cap_\Lambda B_\alpha = [0, 1]$$

   (b) Find $\cup_\Lambda B_\alpha$.

   $$\cup_\Lambda B_\alpha = (-1, 2)$$

   (c) For each value of x below, find a value of α for which $x \in B_\alpha$. If not possible, say so.

      i. $x = -\frac{2}{\pi}$

      Various, so long as $\alpha \leq \frac{1}{3}$

      ii. $x = -\frac{2}{7}$

      Any legitimate α will work.

      iii. $x = 1.02$

      Any legitimate α will work.

      iv. $x = 3\frac{1}{2}$

      This is not possible.

6. (a) *[4 pts]* State the Archimedean Property in mathematically precise language.

   Let r be a real number. Then there exists an integer n for which $n > r$.

   (b) *[4 pts]* Let Λ be an indexing set for the collection $\{B_\alpha\}_{\alpha\in\Lambda}$. State the definition of $\cup_\Lambda B_\alpha$ in mathematically precise language.

   $\cup_\Lambda B_\alpha = \{x \mid \textit{there exists } \alpha \in \Lambda \textit{ with } x \in B_\alpha\}$

7. *[10 pts - 2 each]* Classify each statement below as always, sometimes, or never true. More specifically, these terms mean:

   - Always true: there do not exist counterexamples to the claim.
   - Sometimes true: there exist both examples of and counterexamples to the claim.
   - Never true: There do not exist any examples of the claim.

   (a) The sum of two even integers is even.

   always true   *sometimes true*   *never true*

   (b) The sum of two odd integers is odd.

   always true   *sometimes true*   **never true**

   (c) The product of two rational numbers is rational.

   always true   *sometimes true*   *never true*

   (d) The product of two irrational numbers is irrational.

   always true   **sometimes true**   *never true*

   (e) The quotient of two even integers is even.

   always true   **sometimes true**   *never true*

8. *[9 pts - 3 each]* For each statement type below, tell what you would assume and what you would try to show in writing the indicated type of proof.

   (a) "If P, then Q" by contradiction

   Assume P and $\sim Q$. Try to show any contradiction.

   (b) "If P, then Q or S" by contrapositive

   Assume $\sim Q$ and $\sim S$. Try to show $\sim P$.

   (c) "If P, then Q or S" NOT by contrapositive

   Assume P and $\sim Q$. Try to show S.

9. *[15 pts]* Take-Home Problem due Monday: DO NOT collaborate on this problem. You may refer to your notes, text, and homework, but do NOT refer to any other resources, including the Internet.

Let A and B be sets. Prove that $A \subseteq B$ if and only if $A \cup C \subseteq B \cup C$ for all sets C.

$\Longrightarrow$: *Let A and B be sets, and suppose that $A \subseteq B$. Let C be any set, and let $x \in A \cup C$. By definition of union, $x \in A$ or $x \in C$.*
Case 1: If $x \in A$, then by definition of subset and the assumption that $A \subseteq B$, we have $x \in B$. Then it is true that $x \in B$ or $x \in C$, so $x \in B \cup C$, as desired.
Case 2: If $x \in C$, then it is true that $x \in B$ or $x \in C$, so again $x \in B \cup C$, as desired.
Therefore, $A \cup C \subseteq B \cup C$. (Because C was arbitrary, this is true for all sets C.)

$\Longleftarrow$: *(Direct option) Suppose that $A \cup C \subseteq B \cup C$ for all sets C. Then it is certainly true for $C = \emptyset$, so that $A \cup \emptyset \subseteq B \cup \emptyset$. We prove that $A \cup \emptyset = A$:*
Let $x \in A \cup \emptyset$. By definition of union, $x \in A$ or $x \in \emptyset$. However, $x \in \emptyset$ is impossible, so we must have $x \in A$, proving that $A \cup \emptyset \subseteq A$. Now let $x \in A$. Then it is certainly true that $x \in A$ or $x \in \emptyset$ (An "or" statement is true when just one component is true), so $x \in A \cup \emptyset$, proving that $A \subseteq A \cup \emptyset$. Therefore, $A \cup \emptyset = A$.
Similarly, $B \cup \emptyset = B$.
Thus, the statement $A \cup \emptyset \subseteq B \cup \emptyset$ from earlier becomes $A \subseteq B$, as desired.

$\Longleftarrow$: *(Proof by contrapositive option) Suppose that $A \not\subseteq B$. (NTS: There exists a set C for which $A \cup C \not\subseteq B \cup C$.) Let $C = \emptyset$. We prove that $A \cup \emptyset = A$:*
Let $x \in A \cup \emptyset$. By definition of union, $x \in A$ or $x \in \emptyset$. However, $x \in \emptyset$ is impossible, so we must have $x \in A$, proving that $A \cup \emptyset \subseteq A$. Now let $x \in A$. Then it is certainly true that $x \in A$ or $x \in \emptyset$ (An "or" statement is true when just one component is true), so $x \in A \cup \emptyset$, proving that $A \subseteq A \cup \emptyset$. Therefore, $A \cup \emptyset = A$.
Similarly, $B \cup \emptyset = B$.
Because $A \cup \emptyset = A$ and $B \cup \emptyset = B$, we see that $A \cup \emptyset \not\subseteq B \cup \emptyset$, so there exists a set C for which $A \cup C \not\subseteq B \cup C$, as desired.

(The second half of the proof, regardless of your approach, is a perfect opportunity either to use Lemma 1 from this past week in class, or to create your own lemma. If you wanted to do the latter, stylistically you would prove "Lemma: Let X be a set. Then $X = X \cup \emptyset$." before anything else, THEN launch into the entire proof – both directions – about A, B, and C. It would also be acceptable style-wise to prove the $\Longrightarrow$ direction, then insert the proof of the lemma, and finally finish with the $\Longleftarrow$ direction.)

My signature indicates that I have not collaborated with any other person on this problem, nor have I referred to any other materials, printed, online, or otherwise, besides my notes, text, and homework.

Signed, ____