2. Consider $\mathbb{R}^2 \setminus \{(0, 0)\}$ with these operations:

\[
(a, b) + (c, d) = (ac/2, bd/3)
\]
\[
\alpha(a, b) = (\alpha a, \alpha b)
\]

Determine whether this structure satisfies or fails each of the FIVE conditions pertaining to vector addition in the definition of a vector space. Prove your claims rigorously.

\[C2 \text{ FAILS: Let } (a, b), (c, d) \in \mathbb{R}^2 \setminus \{(0, 0)\}. \text{ Then } (a, b) + (c, d) \in \mathbb{R}^2 \setminus \{(0, 0)\} \text{ is NOT true in general: counterexample when we use } (1, 0) \text{ and } (0, 1). \text{ (Their sum is } (0, 0), \text{ which does not belong.)}
\]

\[A1 \text{ SATISFIED: Let } (a, b), (c, d) \in \mathbb{R}^2 \setminus \{(0, 0)\}.
\]
\[
(a, b) + (c, d) = (ac/2, bd/3) = (ca/2, db/3) = (c, d) + (a, b), \text{ as desired.}
\]

\[A2 \text{ SATISFIED: Let } (a, b), (c, d), (e, f) \in \mathbb{R}^2 \setminus \{(0, 0)\}.
\]
\[
[(a, b) + (c, d)] + (e, f) = (ac/2, bd/3) + (e, f) = (ace/4, bdf/9), \text{ and}
\]
\[
(a, b) + [(c, d) + (e, f)] = (a, c) + (ce/2, df/3) = (ace/4, bdf/9).
\]

These are equal, as desired.

\[A3 \text{ SATISFIED: The vector } (2, 3) \in \mathbb{R}^2 \setminus \{(0, 0)\}, \text{ and for all } (a, b) \in \mathbb{R}^2 \setminus \{(0, 0)\}, (a, b) + (2, 3) = (2a/2, 3b/3) = (a, b), \text{ as desired.}
\]

\[A4 \text{ FAILS: Let } (a, b) \in \mathbb{R}^2 \setminus \{(0, 0)\}. \text{ We want to solve for an inverse } (x, y) \text{ where } (a, b) + (x, y) = (2, 3), \text{ our identity vector. It is not true in general that } (x, y) \text{ can be found: counterexample when } (a, b) = (1, 0) \text{ (for then } (1, 0) + (x, y) = (1x/2, 0y/3), \text{ and we’d need } 1x/2 = 2 \text{ and } 0y/3 = 3, \text{ yet the second equality has no solutions).}
\]

\[C1 \text{ FAILS: Let } \alpha \in \mathbb{R} \text{ and } (a, b) \in \mathbb{R}^2 \setminus \{(0, 0)\}. \text{ Then } \alpha(a, b) = (\alpha a, \alpha b) \in \mathbb{R}^2 \setminus \{(0, 0)\} \text{ is not in general true - counterexample when } \alpha = 0 \text{ and } a = b = 1.
\]

\[A5 \text{ FAILS: Let } \alpha \in \mathbb{R} \text{ and let } (a, b), (c, d) \in \mathbb{R}^2.
\]
\[
\alpha[(a, b) + (c, d)] = \alpha(ac/2, bd/3) = (\alpha ac/2, \alpha bd/3), \text{ and}
\]
\[
\alpha(a, b) + \alpha(c, d) = (\alpha a, \alpha b) + (\alpha c, \alpha d) = (\alpha^2 ac/2, \alpha^2 bd/3).
\]

These are not in general equal - counterexample when } \alpha = 2 \text{ and } a = b = c = d = 1.

\[A6 \text{ FAILS: Let } \alpha, \beta \in \mathbb{R} \text{ and let } (a, b) \in \mathbb{R}^2.
\]
\[
(\alpha + \beta)(a, b) = ((\alpha + \beta)a, (\alpha + \beta)b), \text{ and}
\]
\[
\alpha(a, b) + \beta(a, b) = (\alpha a, \alpha b) + (\beta a, \beta b) = (\alpha \beta a^2/2, \alpha \beta b^2/3).
\]

These are not in general equal - counterexample when } \alpha = 0 \text{ and } \beta = a = b = 1.

\[A7 \text{ SATISFIED: Let } \alpha, \beta \in \mathbb{R} \text{ and let } (a, b) \in \mathbb{R}^2.
\]
\[
(\alpha \beta)(a, b) = (\alpha \beta a, \alpha \beta b) = (\alpha \beta a, \beta b) = \alpha(\beta(a, b)), \text{ as desired.}
\]

\[A8 \text{ SATISFIED: Let } (a, b) \in \mathbb{R}^2. \text{ Then } 1(a, b) = (a, b), \text{ as desired.}
\]