Key

Math 320 - Dr. Miller - Exam #2, Fall 2015 - Friday, Oct. 23, 2015 - **In-Class Portion**

This portion of the exam is worth 75 points.

1. *[10 pts]* Complete ONE of the following tasks; label as Problem A, B, or C:
   (a) Clearly state the condition(s) under which a single congruence in two variables lacks solutions, then formally prove your claim.
   (b) Clearly state the condition(s) under which a system of two linear congruences in one variable lacks solutions, then formally prove your claim.
   (c) Clearly state the condition(s) under which a single linear diophantine equation in two variables lacks solutions, then formally prove your claim.

2. *[12 pts - 4 each]* Solve the following, if possible; if not, say so. Show clear work, and give answers as least nonnegative residues:
   (a) $19x \equiv 15 \ mod\ 8$
   (b) $21x \equiv 14 \ mod\ 9$
   (c) $12x \equiv 6 \ mod\ 14$

3. (a) *[8 pts]* Clearly state the definition of multiplicative inverse mod n, then find it for 197 mod 16, giving your answer as a least nonnegative residue.
   (b) *[4 pts]* Give an example of a positive, composite integer that would not have a multiplicative inverse mod 30, telling why.

4. (a) *[9 pts]* Demonstrate appropriate divisibility tests to determine whether the 5-digit number 41,470 is divisible by each of 3, 11, and/or 7.
   (b) *[4 pts]* Give an example of a 6-digit number for which the divisibility test for 8 actually can be done mentally; tell in a sentence how.
   (c) *[4 pts]* Find all possible 4-digit numbers of the form c23d that are divisible by 12. Show work if needed.

5. *[8 pts]* Prove a divisibility test for 9.

6. *[8 pts]* Eleanor bought doughnuts and coffee totaling $7.25. Doughnuts are 50 cents apiece; coffee is 75 cents a cup. How many of each is it possible that she bought? Show work, and give answers meaningfully.

7. *[8 pts]* Prove that the diophantine equation $3x^2 - 5y^2 = 2$ has no solutions.

Math 320 - Exam #2 Solutions - F, 10/23/15
In-Class

-1 no notice of ctp.

1.A. (I forgot to specify "linear" here, but it's okay if you assumed it, as well as if not. Here's the linear proof) (Nonlinear will not differ meaningfully.) The linear congruence $ax + by \equiv c \bmod m$ (where $a, b, c \in \mathbb{Z}$ and $m \in \mathbb{Z}^+$) does not have solutions if $\gcd(a, b, m) \nmid c$.

-3 linear in 1 variable.

Proof (by ctp) - Suppose there exists a solution (x_0, y_0) and let $d = (a, b, m)$. Then $a \cdot x_0 + by_0 \equiv c \bmod m$ implies that $a x_0 + by_0 + mk = c$ for some $k \in \mathbb{Z}$. Now $d \mid a, b, m$, so it divides the linear combination shown: that is, $d \mid c$.
By ctp, if $d \nmid c$, then the solution cannot exist.

B. The system $x \equiv b_1 \bmod m_1$, $x \equiv b_2 \bmod m_2$ (where $b_1, b_2 \in \mathbb{Z}$ and $m_1, m_2 \in \mathbb{Z}^+$) does not have solutions if $\gcd(m_1, m_2) \nmid b_1 - b_2$.

Proof (by ctp) - Suppose there exists a solution $x_0 \equiv b_1 \bmod m_1$, $x_0 \equiv b_2 \bmod m_2$. Then $x_0 = b_1 + m_1 k = b_2 + m_2 l$ for some $k, l \in \mathbb{Z}$, whence $b_1 - b_2 = m_2 l - m_1 k$. Now $(m_1, m_2) \mid m_1, m_2$, so it divides the linear combination shown: $(m_1, m_2) \mid b_1 - b_2$.

By ctp, if $(m_1, m_2) \nmid b_1 - b_2$, then the solution cannot exist.

c. The linear diophantine equation $ax + by = c$ (where $a, b, c \in \mathbb{Z}^+$) has no solutions if $\gcd(a,b) \nmid c$.

Proof (by ctp) - assume that there exists a solution (x_0, y_0),

so $ax_0 + by_0 = c$.

Now $(a,b) \mid a, b$, so it divides the linear combination shown: that is, $(a,b) \mid c$.

By ctp, if $(a,b) \nmid c$, then the solution cannot exist.

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2. a. (Various) $19x \equiv 15 \mod 8$ or $19 \times 3 = 57 \equiv 1$.

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$16 \equiv 0 \Rightarrow$ $3x \equiv 15 \mod 8$

$\div 3 \quad \div 3 \quad \div \gcd$

$$\boxed{x \equiv 5 \mod 8}$$

b. (Various) $\gcd(21, 9) \nmid 14$ so $\boxed{\text{no solns}}$

4

OR $21x \equiv 14 \mod 9$

$\div 7 \quad \div 7 \quad \div \gcd$

$3x \equiv 2 \mod 9$

$\times 3 \quad \times 3$

$0 \equiv 6 \mod 9$ no solns.

$2x \equiv 1$ to straight to $x \equiv 8$ is ok.

c. (Various) $12x \equiv 6 \mod 14$

4

$\div 6 \quad \div 6 \quad \div \gcd$

$2x \equiv 1 \mod 7$

$\times 4 \quad \times 4$

$x \equiv 4 \mod 7$

so $\boxed{x \equiv 4, 11 \mod 14}$

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$\Rightarrow (a,n)=1$ as part of defn.

3. a. 8 (Various) 4 Integers a and b are multiplicative inverses mod n, where $n \in \mathbb{Z}^+$, if $ab \equiv 1 \mod n$.

OR Given $a, n \in \mathbb{Z}$ with $n \geq 1$, x is a multiplicative inverse of a mod n if $ax \equiv 1 \mod n$.

4

$$197 \cdot \underline{\ ?\ } \equiv 1 \mod 16$$

$192 \equiv 0 \Rightarrow$ $$5x \equiv 1 \mod 16$$

$$5 \cdot (-3) = -15 \equiv 1 \mod 16$$

-1 -3

so $197^{-1} = -3 \equiv \boxed{13 \mod 16}$

b. 4 (Various) 12 has no multiplicative inverse mod 30 because $(12, 30) \neq 1$.

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4. a. 9 $4+1+4+7+0=16$ $3 \nmid 16$, so $3 \nmid 41{,}470$

$4+4+0=8$, $1+7=8$, $8-8=0$ $11 \mid 0$, so $11 \mid 41{,}470$

$$\begin{array}{r} 41\cancel{47}\cancel{0} \\ -14 \\ \hline 400 \end{array}$$

$7 \nmid 400$, so $7 \nmid 41{,}470$.

b. 4 543,008 is divisible by 8: the final 3-digit portion is easy to recognize as a multiple of 8 (Various)

c. Need $4 \mid c23d$, so $d = 2, 6$

4 Need $3 \mid c+2+3+d$

$d = 2 \Rightarrow c = 2, 5, 8$

$d = 6 \Rightarrow c = 1, 4, 7$

2,232
5,232
8,232
and
1,236
4,236
7,236

8 5. Test I: $9 \mid N$ iff 9 divides the sum of N's digits.

-1 no a_i constraints

Proof - Let $N = \sum_{i=0}^{n} 10^i a_i$ where each $a_i \in \{0,1,\ldots,9\}$

-4 no $\equiv$ invoked.

Then $10 \equiv 1 \mod 9$ implies $10^i \equiv 1 \mod 9$ for all integers $i \geq 0$.

Thus $N \equiv \sum_{i=0}^{n} a_i \mod 9$, whence

$N \equiv 0$ iff $\sum_{i=0}^{n} a_i \equiv 0$, i.e.

$9 \mid N$ iff 9 divides the sum of N's digits.

Test II: "Drop and add"

(-1) no b constraints

Proof - Let $N = 10a + b$ where $a \in \mathbb{Z}^+ \cup \{0\}$ and $b \in \{0, 1, \ldots, 9\}$.

$$9 \mid N \quad \text{iff} \quad 9 \mid 10a + b$$
$$\text{iff} \quad 9 \mid 10a + b - 9a$$
$$\text{iff} \quad 9 \mid a + b,$$

which is the expression created by "drop and add".

(-5) (8)

6. $50x + 75y = 725$

(4) one soln w/o general approach

(-3) one soln w/ general approach.

Method I: One soln is $y = 1$, $x = 13$

Trade 2 cups of coffee for 3 doughnuts repeatedly:

$y = 3$, $x = 10$
$y = 5$, $x = 7$
$y = 7$, $x = 4$
$y = 9$, $x = 1$

Answers:
1 donut + 9 coffees or (1, 9)
4 donuts + 7 coffees or (4, 7)
7 donuts + 5 coffees or (7, 5)
10 donuts + 3 coffees or (10, 3)
13 donuts + 1 coffee or (13, 1)

Method 2: $x = 13, y = 1$ is one soln.

All are of the form $x = 13 + \frac{75}{25}n = 13 + 3n$

$y = 1 - \frac{50}{25}n = 1 - 2n$

for $n \in \mathbb{Z}$.

$13 + 3n \geq 0 \Rightarrow 3n \geq -13 \Rightarrow n \geq -4\frac{1}{3}$

$1 - 2n \geq 0 \Rightarrow -2n \geq -1 \Rightarrow n \leq 0.5$

so $n = 0, -1, -2, -3, -4$

$(13,1)$ $(10,3)$ $(7,5)$ $(4,7)$ $(1,9)$

8 7. $3x^2 - 5y^2 = 2$

-2 confused = with ≡

-4 tried mod 3 and 5 w/o successful interpret.

Mod 5, we have $3x^2 \equiv 2 \mod 5$

or $x^2 \equiv 4 \mod 5$,

which has solutions + so doesn't further our claim of no solutions to the original

Mod 3, however, $-5y^2 \equiv 2 \mod 3$

becomes $y^2 \equiv 2 \mod 3$,

yet none of $y \equiv 0, 1,$ or $2 \mod 3$

satisfy this claim.

Since there exists $n \in \mathbb{Z}^+$ where the congruence mod n has no solutions, we know (by ctp), that the original equation also has none.