The so-called dihedral groups, denoted $D_n$, are permutation groups. By definition, $D_n$ equals the set of symmetries of a regular $n$-gon. Each group $D_n$ is created as follows:

- Draw a regular $n$-gon, and label its vertices $1, 2, \ldots, n$ in a clockwise direction. (Traditionally, one begins by placing vertex 1 in the “12 o’clock position” for an $n$-gon with $n$ odd, and just to the right of the 12 o’clock position for an $n$-gon with $n$ even.)
- Perform any rotation or reflection that demonstrates symmetry of the $n$-gon.
- Use cycle notation to describe what position each numbered vertex “ends up in.”

For example, in $D_3$, we have an equilateral triangle whose vertices are labelled $1, 2, 3$. Rotating this triangle $120^\circ$ clockwise around a center causes vertex 1 to end up where vertex 2 used to be; that is, vertex 1 is in position 2. Similarly, vertex 2 goes to position 3, and vertex 3 goes to position 1. This can be represented by the cycle $(1, 2, 3)$.

1. [1 pt] List the cycle notation for the remaining members of $D_3$.

   \[
   D_3 = \{(1),\ (1, 2),\ (1, 3),\ (2, 3),\ (1, 2, 3),\ (1, 3, 2)\}
   \]

2. Use cycle notation to represent all members of $D_4$. (Not all members are single cycles.)

   \textit{See #3.}

3. [3 pts] It is often useful to identify each member of $D_n$ with the physical motion it describes. Relist $D_4$, describing each permutation verbally in terms of its degrees of clockwise rotation or the axis it reflects through. (A square has a horizontal axis, a vertical one, and two diagonals, the positive sloped one and the negative sloped one.)

   \[
   \begin{array}{c}
   4 \\
   1 \\
   3 \\
   2 \\
   \end{array}
   \]

   (1) is the identity permutation - no motion.
   (1, 2, 3, 4) describes the $90^\circ$ (cw) rotation.
   (1, 3)(2, 4) describes the $180^\circ$ rotation.
   (1, 4, 3, 2) describes the $270^\circ$ (cw) rotation.
   (1, 2)(3, 4) describes reflection in the horizontal axis.
   (1, 4)(2, 3) describes reflection in the vertical axis.
   (1, 3) describes reflection in the positive-sloped diagonal.
   (2, 4) describes reflection in the negative-sloped diagonal.
4. [4 pts] List the members of $D_5$ and $D_6$. Are these groups abelian? Are they cyclic?

$D_5$ contains the identity and 4 rotations:

$\begin{align*}
(1), \quad (1,2,3,4,5), \quad (1,3,5,2,4), \quad (1,4,2,5,3), \quad (1,5,4,3,2)
\end{align*}$

and five reflections through axes that join a vertex to the midpoint of the opposite side:

$\begin{align*}
(2,5)(3,4), \quad (1,3)(4,5), \quad (1,5)(2,4), \quad (1,2)(3,5), \quad (1,4)(2,3).
\end{align*}$

$D_6$ contains the identity and 5 rotations:

$\begin{align*}
(1), \quad (1,2,3,4,5,6), \quad (1,3,5)(2,4,6), \quad (1,4)(2,5)(3,6), \quad (1,5,3)(2,6,4), \quad (1,6,5,4,3,2)
\end{align*}$

three reflections through axes that connect opposite vertices:

$\begin{align*}
(2,6)(3,5), \quad (1,3)(4,6), \quad (1,5)(2,4)
\end{align*}$

and three reflections through axes that connect midpoints of opposite sides:

$\begin{align*}
(1,2)(3,6)(4,5), \quad (1,4)(2,3)(5,6), \quad (1,6)(2,5)(3,4).
\end{align*}$

Neither group is abelian. For example, in $D_5$, $(1,2,3,4,5)(2,5)(3,4) = (1,2)(3,5)$ while $(2,5)(3,4)(1,2,3,4,5) = (1,5)(2,4)$ (which clearly aren’t equal). In $D_6$, we can find similar counterexamples. Then because neither group is abelian, it cannot be cyclic. (We know that every cyclic group IS abelian.)

5. [1 pt] Conjecture about the order of each group $D_n$. Try to explain why your conjecture makes sense physically.

$o(D_n) = 2n$ for $n \geq 3$. In a regular $n$-gon, there are $n$ distinct reflectional symmetries (each one rotates the vertex labelled “1” to any of the $n$ vertex positions on the polygon). There are also $n$ axes of reflectional symmetry: if $n$ is odd, each one connects a vertex with the midpoint of the opposite side while if $n$ is even, $n/2$ such axes connect pairs of opposite vertices and another $n/2$ axes connect pairs of midpoints of opposite sides.