You're familiar from childhood with Addition Tables or Times/Multiplication Tables that help us memorize basic number facts:

$+$	1	2	3	4
1	2	3	4	5
2	3	4	5	6
3	4	5	6	7
4	5	6	7	8

OR

$\times$	1	2	3	4
1	1	2	3	4
2	2	4	6	8
3	3	6	9	12
4	4	8	12	16

We read such tables by locating one addend or factor in the outer, lefthand column and the other addend or factor in the outer row across the top of the table. We then follow where their rows and columns overlap in the body of the table to see the resulting sum or product.

For example, here we see how to locate $3+2$ and how to locate 4×1:

$+$	1	2	3	4
1	2	3	4	5
2	3	4	5	6
3	4	$\boxed{5}$	6	7
4	5	6	7	8

Sum

OR

$\times$	1	2	3	4
1	1	2	3	4
2	2	4	6	8
3	3	6	9	12
4	$\boxed{4}$	8	12	16

Product

Such details seem trivial now, they do apply to abstract algebra: As children, we learn to use such tables to help memorization, not so much to help understand the actual operations of addition and multiplication and what they mean. But in abstract algebra, where operations might not be the familiar ones whose results we've memorized, a table can help us determine the results of applying an unusual operation. Such tables are called Cayley tables, after the 19th century British mathematician Arthur Cayley who was one of the first people to study abstract algebra.

Below is a table for a binary operation on a very small set. There is no formula given for the operation, nothing like $a * b = 2ab + 1$, but we can still determine the results of applying the operation just like above:

To find the "product" $\boxed{?} * \underline{?}$ of any two elements in the set:

- Find the first element, $\boxed{?}$, in the column of row-labels outside the lefthand edge of the table.
- Find the second element, $\underline{?}$, in the row of column-headings above the very top of the table.
- Where $\boxed{?}$'s row and $\underline{?}$'s column overlap in the body of the table is the value of $\boxed{?} * \underline{?}$.

For example, here is how to find $z * y$ in this table:

$*$	x	y	z	w
x	z	w	x	z
y	y	w	y	w
z	z	$\boxed{w}$	z	w
w	y	x	w	y

$z * y = w$

Use the table (clean copy below) to practice on your own: find $x * y$, $y * x$, $y * z$, $w * x$, and $w * w$.

$*$	x	y	z	w
x	z	w	x	z
y	y	w	y	w
z	z	w	z	w
w	x	x	w	z

Now try this challenge: find $(x * x) * (w * y)$ and find $(z * y) * w$ versus $z * (y * w)$.

Childhood tables and Cayley tables help us recognize important patterns and properties as well:

Closure: In the children's Multiplication Table starting this lesson, you see that some of the numbers in the body of the table are NOT among those shown as the row- and column-labels around the outside the table. That means that if I am using that multiplication table to define on operation on the set $S = \{1, 2, 3, 4\}$, S is not closed. Observe how our abstract Cayley table shows us a set that IS closed under the operation.

$\times$	1	2	3	4
1	1	2	3	4
2	2	4	6	8
3	3	6	9	12
4	4	8	12	16

not in
$S = \{1, 2, 3, 4\}$

$*$	x	y	z	w
x	z	w	x	z
y	y	w	y	w
z	z	w	z	w
w	x	x	w	z

all entries belong to
$S = \{x, y, z, w\}$

Commutativity: We see this very clearly in the two children's tables. Every time we add or multiply two numbers in one order $a + b$ or $a \times b$, reading from a's row and b's column, we find the same result for $b + a$ or $b \times a$, which read from <u>b's row</u> and <u>a's column</u>. A visual short-cut for this thinking about individual rows and columns trading behaviors is simply that the two tables show reflectional symmetry across the diagonal that runs from $1 + 1$ or 1×1 down to the $4 + 4$ or 4×4 entry.

$\times$	1	2	3	4
1	1	2	3	4
2	2	4	6	8
3	3	6	9	12
4	4	8	12	16

= = =

etc.

Has reflectional symmetry, so $\times$ <u>is</u> commutative.

This short-cut can be applied to any Cayley table, so long as the row- and column-headings read in exactly the same order going down the column outside the left of the table and going left-to-right in the row outside the top of the table.

Notice that our Cayley table is definitely NOT symmetric, so $*$ is not commutative. If asked, we can easily offer some concrete counterexamples.

$*$	x	y	z	w
x	z	w	x	z
y	y	w	y	w
z	z	w	z	w
w	x	x	w	z

$\neq$

$y * x = y$
but
$x * y = w$
these aren't equal,
so $*$ is <u>NOT</u> commutative.

Identity Element: These are also easy to spot visually in a Cayley table - since an identity leaves all elements in the set unchanged, the row an identity labels will be identical to the original listing at the top of the table and the column for which it is the heading will also be identical to the original left-side outer column. Notice how this is true for the number 1 in the children's Times Table:

$\times$	1	2	3	4
1	1	2	3	4
2	2	4	6	8
3	3	6	9	12
4	4	8	12	16

matches these labels

matches these labels

The identity behavior is NOT shown by any elements in our earlier Cayley table, though z is certainly giving a good effort. z's column shows the correct behavior for z to be an identity element, but z's row wrecks its chances for such fame.

$*$	x	y	z	w
x	z	w	x	z
y	y	w	y	w
z	z	w	z	w
w	x	x	w	z

doesn't match

matches!

Inverse Elements: We can find these in Cayley tables also. **If** the set has an identity, we just look for that element as an entry in the BODY of the table. We can then reconstruct which two elements created that entry, and whether they do so in reverse order as well. Here's a new table to check this out:

$\triangle$	x	y	z	w
x	z	w	x	z
y	y	w	y	w
z	x	y	z	w
w	z	x	w	y

z is the identity now.

$x * x = z$, so x is its own inverse

Also $w * x = z$ and $x * w = z$, so w and x are inverses.

z is its own inverse (no surprise)

y has no inverse since $y * \square$ or $\triangle * y$ never create z in the body of the table.

Associativity: It's much messier to determine associativity from a Cayley table, and quite often is not worth the trouble. The issue is that associativity requires us to think about the interactions of **three** elements from our set, yet the Cayley table can only show how **two** interact at a time. To check for associativity, one either needs to build a three-dimensional Cayley array, or else check examples by hand one at a time (as you did in the first Cayley table exercise on p.1) to exhaust all possibilities. Both options are time-consuming, and not often followed-through on.

However, there is **ONE** visual short-cut we can used, based on our understanding of logic:

Notice in the table at the bottom of the last page, that x apparently has two inverses. We proved recently that inverse elements ARE unique, so what's going on here? In fact, the statement we proved wasn't a static "Inverse elements are unique" statement, but rather a conditional statement, "If $*$ is associative, then inverses are unique."

We proved that conditional statement, so it is undeniably true. What the previous Cayley table shows is that we have a FALSE CONCLUSION to the true conditional we proved. This can be meaningfully interpreted in two ways that you all have learned previously:

1) The only row in a truth table for $p \rightarrow q$ that has $p \rightarrow q$ TRUE while q is false is the (typically very bottom row) that has $p = false$. Therefore, we conclude that the hypothesis of our conditional statement above is FALSE: $*$ is not associative.

2) We know that a conditional statement and its contrapositive are logically equivalent, meaning that they have the same truth values. Since "If $*$ is associative, then inverses are unique" is true, the contrapositive is also true, and it says "If inverses are NOT unique, then $*$ is NOT associative.' So this thinking also tells us that $*$ in our table can't be associative.

So this uniqueness result does give a way to just look at a 2-dimensional Cayley table and know that it does NOT shows an associative operation. However, it's not possible to use this visual short-cut very much of the time:

(a) If a table doesn't show an identity in the first place, inverses won't exist either, so we'll not be observing anything about them, uniqueness or otherwise.

(b) Even if inverses exist and are unique, that tells us nothing about associativity. Having a TRUE conclusion to a TRUE conditional statement happens for TWO rows in a truth table, and for one of them the hypothesis is true while for the other it is false. So we can't draw any conclusions about our hypothesis of associativity. You could also recognize that the wishful statement "If inverses are true, then $*$ ...[does something]" corresponds to the CONVERSE of our proven statement, which is not logically equivalent to our original statement.

The bottom line is that most of the time, if we absolutely must check associativity from a Cayley table, we're stuck checking all expressions $a * b * c$ (where some elements could be equal), and if our set has cardinality n, that's n^3 expressions .