1. [21 pts - 3 each] Prove each of the following claims by your choice of direct proof or proof by contraposition. Clearly state all assumptions, and use rigorous logic (i.e., don’t skip steps).

(a) If \(x\) and \(y\) are both odd, then \(x + y\) is even. \((x, y\) are integers.)

(Direct Proof.) Let \(x\) and \(y\) be odd integers. Then by definition, there exist integers \(m\) and \(n\) such that \(x = 2m + 1\) and \(y = 2n + 1\). The sum \(x + y = (2m + 1) + (2n + 1) = 2(m + n + 1)\) is even because \(m + n + 1\) is the sum of integers and therefore an integer itself.

(b) If \(xy\) is even, then at least one of \(x\) or \(y\) is even. \((x, y\) are integers.)

(Proof by contrapositive.) Suppose it is not the case that at least one of \(x\) or \(y\) is even; that is, suppose that both \(x\) and \(y\) are odd. Then by definition, there exist integers \(m\) and \(n\) such that \(x = 2m + 1\) and \(y = 2n + 1\). The product \(xy = (2m + 1)(2n + 1) = 2(2mn + m + n) + 1\) is odd because \(2mn + m + n\) is the sum and product of integers and therefore an integer itself. Therefore, by contrapositive, if \(xy\) is even, then at least one of \(x\) or \(y\) is even.

(c) If \(x + y\) is odd, then \(x\) and \(y\) have different parity. \((x, y\) are integers.)

(Proof by contrapositive.) Suppose \(x\) and \(y\) have the same parity; that is, suppose that either both are odd or both are even. If both are odd, then by definition, there exist integers \(m\) and \(n\) such that \(x = 2m + 1\) and \(y = 2n + 1\). Then the sum \(x + y = (2m + 1) + (2n + 1) = 2(m + n + 1)\) is even because \(m + n + 1\) is the sum of integers and therefore an integer itself. If both are even, then by definition, there exist integers \(k\) and \(\ell\) such that \(x = 2k\) and \(y = 2\ell\). Then again the sum \(x + y = 2k + 2\ell = 2(k + \ell)\) is even because \(k + \ell\) is the sum of integers and therefore an integer itself. Therefore, by contrapositive, if \(x + y\) is odd, then \(x\) and \(y\) have different parity.

(d) The sum of two rational numbers is rational. (Carefully identify hypothesis and conclusion first, introducing variables.)

(Direct Proof.) Suppose that \(p\) and \(q\) are rational numbers. Then there exist integers \(a\) and \(b\) with \(b \neq 0\) for which \(p = \frac{a}{b}\), and there exist integers \(c\) and \(d\) with \(d \neq 0\) for which \(q = \frac{c}{d}\). The sum \(p + q = \frac{ad + bc}{bd}\) is rational because the numerator is an integer - being the sum and product of integers - and the denominator is a non-zero integer, being the product of non-zero integers.

(e) The quotient of a rational number and a non-zero rational number is rational.

(Direct Proof.) Suppose that \(p\) and \(q\) are rational numbers with \(q \neq 0\). Then there exist integers \(a\) and \(b\) with \(b \neq 0\) for which \(p = \frac{a}{b}\), and there exist integers \(c\) and \(d\) with \(d \neq 0\) for which \(q = \frac{c}{d}\). Furthermore, because \(q \neq 0\), we must have \(c \neq 0\) also. The quotient \(p/q = \frac{ad}{bd}\) is rational because the numerator is an integer - being the product of integers - and the denominator is a non-zero integer, being the product of non-zero integers.

(f) If \(p\) is rational and \(q\) is irrational, then \(p + q\) is irrational. \((p \text{ and } q\) are real numbers.)

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(Proof by contrapositive.) Suppose that \(p+q\) is rational. \((\text{We need to show that it is NOT the case that } p\text{ is rational and } q\text{ is irrational. That is, we must show that } p\text{ is irrational or } q\text{ is rational. This is an “or” conclusion, so assume additionally that one part of the conclusion is NOT true and try to show that the other part of the conclusion IS true.})\) Suppose also that \(p\) is rational. Thus, by definition we have integers \(a\) and \(b\) with \(b \neq 0\) for which \(p = \frac{a}{b}\), so that \(-p\) is rational, for it can be written as \(-\frac{a}{b}\), whose numerator is an integer and denominator is a non-zero integer. Then by Problem #1d, the sum of the rational numbers \(p + q\) and \(-p\) is rational, but this sum is \(q\), making \(q\) rational, as desired.
If \( a < b \) and \( c < d \), then \( a + c < b + d \). (\( a, b, c, \) and \( d \) are real numbers. Use the definition of \(<\), not any algebraic properties of it.)

(Direct Proof.) Suppose that \( a, b, c, \) and \( d \) are real numbers with \( a < b \) and \( c < d \). Then by definition there exist positive real numbers \( g \) and \( h \) for which \( a + g = b \) and \( c + h = d \). By algebra, \( a + g + c + h = b + d \), or \((a + c) + (g + h) = b + d \). Because \( g + h \) is positive, being the sum of positive real numbers, the definition of \(<\) confirms that \( a + c < b + d \).

2. [13 pts - 1-2 each] Below are several somewhat informally worded statements: they represent many ideas I see students try to use in proofs, but NOT ALL are true! Formally PROVE or DISPROVE each statement, using any meaningful proof technique from our course so far. (Introduce variables if needed, and DON’T SKIP STEPS.)

(a) An even number minus an even number is even.

Proof. Let \( x \) and \( y \) be even numbers. Then there exist integers \( m \) and \( n \) for which \( x = 2m \) and \( y = 2n \). The difference \( x - y = 2n - 2m = 2(n - m) \), which is even because the \( n - m \) is the difference of integers and so is an integer also.

(b) An even number divided by an even number is even.

Disproof. Consider the numbers \( 12 \) and \( 4 \). Both are even because \( 12 = 2 \times 6 \) and \( 4 = 2 \times 2 \). Yet \( 12 \div 4 = 3 \), which is not even because \( 3 \) cannot be written as the product of \( 2 \) and an integer.

(c) An odd number minus an odd number is odd.

Disproof. Consider the numbers \( 5 \) and \( 3 \). Both are odd, for \( 5 = 2(2) + 1 \) and \( 3 = 2(1) + 1 \), and both \( 2 \) and \( 1 \) are integers. Yet \( 5 - 3 = 2 \) is not odd, for it cannot be written in the form \( 2n + 1 \) for any integer \( n \).

(d) An odd number divided by an odd number is odd.

Disproof. Consider the numbers \( 5 \) and \( 3 \). Both are odd, for \( 5 = 2(2) + 1 \) and \( 3 = 2(1) + 1 \), and both \( 2 \) and \( 1 \) are integers. Yet \( 5 \div 3 = \frac{5}{3} \) is not odd, for it cannot be written in the form \( 2n + 1 \) for any integer \( n \).

(e) A rational number minus a rational number is rational.

Proof. Let \( p \) and \( q \) be rational. Then there exist integers \( a, b, c, \) and \( d \) with \( b \neq 0 \) and \( d \neq 0 \) for which \( p = a/b \) and \( q = c/d \). Then \( p - q = \frac{ad - bc}{bd} \), which is rational: the numerator \( ad - bc \) is an integer because it is the product and difference of integers, and the denominator \( bd \) is a nonzero integer because it is the product of nonzero integers.

(f) A rational number times a rational number is rational.

Proof. Let \( p \) and \( q \) be rational. Then there exist integers \( a, b, c, \) and \( d \) with \( b \neq 0 \) and \( d \neq 0 \) for which \( p = a/b \) and \( q = c/d \). Then \( pq = \frac{ac}{bd} \), which is rational: the numerator \( ac \) is an integer because it is the product of integers, and the denominator \( bd \) is a nonzero integer because it is the product of nonzero integers.

(g) An irrational number minus an irrational number is irrational.

Disproof. Consider the numbers \( \sqrt{2} \) and \( \sqrt{2} - 1 \); both are irrational, for we have proved that \( \sqrt{2} \) is irrational, and also by Problem #1d we have that the sum of the irrational number \( \sqrt{2} \) and
the rational number 1 (1 = \frac{1}{1}, the quotient of nonzero integers) is also irrational. However, their difference, 1, is rational.

(h) An irrational number times an irrational number is irrational.

\textit{Disproof. Consider the number }\sqrt{2}; \textit{we have proved that it is irrational. Yet the product }\sqrt{2} \cdot \sqrt{2} = 2 \textit{is rational, for it can be written as } \frac{2}{1}, \textit{a fraction of nonzero integers.}

(i) If real numbers } a \neq b \text{ and } c \neq d, \text{ then } a + c \neq b + d.

\textit{Disproof. Consider } a = c = 2, b = 1, \text{ and } d = 3. \textit{All are real numbers, with } 2 \neq 1 \text{ and } 2 \neq 3, \text{ yet } 2 + 2 = 1 + 3.

(j) If real numbers } a \neq b \text{ and } c \neq d, \text{ then } ac \neq bd.

\textit{Disproof. Consider } a = 2, b = 1, c = 3 \text{ and } d = 6. \textit{All are real numbers, with } 2 \neq 1 \text{ and } 3 \neq 6, \text{ yet } 2 \cdot 3 = 1 \cdot 6.

3. BONUS: [3 pts - 0.5 each] Summarize the potential fallacies exposed through this assignment by completing each sentence below using the words “sum,” “difference,” “product”, and “quotient” (by non-zero divisor) similarly to this example:

The \underline{sum} of two integers is always an integer, but the \underline{quotient}, even when it exists, of two integers might not be an integer.

\textit{Answer: The sum, difference, and product of two integers is always an integer, but the quotient, even when it exists, of two integers might not be an integer.}

(a) The \underline{sum, difference, and product} of two even integers is always an even integer, but the \underline{quotient, even when it exists}, of two even integers might not be an even integer.

(b) The \underline{product} of two odd integers is always an odd integer, but the \underline{sum, difference, and quotient, even when it exists}, of two odd integers might not be an odd integer.

(c) The \underline{sum, difference, product, and quotient (when it exists)} of two rational numbers is always a rational number.

(d) The \underline{sum, difference, product, and quotient (when it exists)} of two irrational numbers might not be an irrational number.

(e) If } a < b \text{ and } c < d, \text{ then the } \underline{sum and difference} \text{ of } a \text{ and } c \text{ is always less than that of } b \text{ and } d, \text{ but the } \underline{product and quotient (when it exists)} \text{ might not be.}

(f) If } a \neq b \text{ and } c \neq d, \text{ then the } \underline{sum, difference, product, and quotient (when it exists)} \text{ of } a \text{ and } c \text{ might not be unequal to that of } b \text{ and } d.