1. \[4 \text{ pts}\] Prove carefully, using correct notation throughout: Let \(A\) and \(B\) be sets. Then \(A \subseteq B\) if and only if \(\mathcal{P}(A) \subseteq \mathcal{P}(B)\).

**Proof:** \((\Rightarrow):\) Let \(A\) and \(B\) be sets with \(A \subseteq B\), and let \(X \in \mathcal{P}(A)\). By definition of power set, we know that \(X \subseteq A\), and by transitivity, \(X \subseteq B\), so \(X \in \mathcal{P}(B)\) again by definition of power set. Thus, \(\mathcal{P}(A) \subseteq \mathcal{P}(B)\).

\((\Leftarrow):\) Let \(A\) and \(B\) be sets with \(\mathcal{P}(A) \subseteq \mathcal{P}(B)\), and let \(x \in A\). Consider the set \(X = \{x\}\). By definition of subset, \(X \subseteq A\), so that by definition of power set \(X \in \mathcal{P}(A)\). But now by our assumption, \(X \in \mathcal{P}(B)\) also, so that \(X \subseteq B\). The definition of subset requires that every member of \(X\) be a member of \(B\), so we see that \(x \in B\), as desired: \(A \subseteq B\).

2. \[4 \text{ pts}\] Let \(A\) and \(B\) be sets, and consider \(\mathcal{P}(A \setminus B)\) and \(\mathcal{P}(A) \setminus \mathcal{P}(B)\). Prove that neither of these sets is a subset of the other.

Let \(A = \{1, 2\}\) and \(B = \{1\}\). Then \(A \setminus B = \{2\}\), so that \(\mathcal{P}(A \setminus B) = \{\emptyset, \{2\}\}\). On the other hand, note that \(\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}\) and \(\mathcal{P}(B) = \{\emptyset, \{1\}\}\), so that \(\mathcal{P}(A) \setminus \mathcal{P}(B) = \{\{2\}, \{1, 2\}\}\).

Then \(\emptyset \in \mathcal{P}(A \setminus B)\) but \(\emptyset \notin \mathcal{P}(A) \setminus \mathcal{P}(B)\), so \(\mathcal{P}(A \setminus B) \not\subseteq \mathcal{P}(A) \setminus \mathcal{P}(B)\). Also, \(\{1, 2\} \in \mathcal{P}(A) \setminus \mathcal{P}(B)\) but \(\{1, 2\} \notin \mathcal{P}(A \setminus B)\), so that \(\mathcal{P}(A) \setminus \mathcal{P}(B) \not\subseteq \mathcal{P}(A \setminus B)\).

3. \[2 \text{ pts}\] Reading problem: In Chapter 3, our author describes the intuition involved in mathematical induction as being related to the idea of finishing an argument by saying “and so on.” She cautions us to beware that this is not always a reliable mode of thinking. State the result for which she claims that “and so on” does correctly describe the mathematical situation at hand, then state the result for which “and so on” is a lie. Moral of the story: Don’t EVER use “and so on” in a proof!

She says that “and so on” does describe the situation correctly for “If \(n \in \mathbb{N}\) and \(S\) is a set with \(n\) elements, then \(\mathcal{P}(S)\) has \(2^n\) elements.” However, this phrase does not correctly describe “For all \(n \in \mathbb{N}\), \(x^2 + x + 41\) is a prime number” (because the earliest counterexample is quite far down the line).

4. \[12 \text{ pts} - 3 \text{ each}\] Prove via mathematical induction:

(a) \(8 \mid (10^n - 2^n)\) for all integers \(n \geq 0\)

When \(n = 0\), it is true that \(8\) divides \(10^0 - 2^0 = 0\) because this difference equals \(8\) times the integer \(0\). Now suppose that \(8 \mid (10^k - 2^k)\) where \(k\) is an integer greater than or equal to \(0\), so that by definition of “divides,” \(10^k - 2^k = 8a\) for some integer \(a\). Consider \(10^{k+1} - 2^{k+1}\). (We need to show that this number is of the form \(8c\)
where \( c \) is an integer.) We have that
\[
10^{k+1} - 2^{k+1} = 10 \cdot 10^k - 2 \cdot 2^k \quad \text{by algebra}
= 10(10^k - 2^k) - 2 \cdot 2^k \quad \text{again by algebra}
= 10(10^k - 2^k) + 10 \cdot 2^k - 2 \cdot 2^k \quad \text{by the Distributive Property}
= 10(10^k - 2^k) + 2^k \cdot 8 \quad \text{again by the Distributive Property}
= 10 \cdot 8a + 2^k \cdot 8 \quad \text{by substitution from our assumption}
= 8(10a + 2^k) \quad \text{by algebra once more}
\]
In this last expression, \( 10a + 2^k \) is an integer because it involves the sum, product, and WHOLE NUMBER POWER of integers and therefore is itself an integer. Thus, by definition, \( 8 \mid 10^{k+1} - 2^{k+1} \), and by PMI, it is true that \( 8 \mid (10^n - 2^n) \) for all integers \( n \geq 0 \).

(b) \( 15 \mid (14^{2n-1} + 1) \) for all integers \( n \geq ? \), where you determine the correct “base case”

This statement is true for all integers \( n \geq 1 \). When \( n = 1 \), it is true that 15 divides \( 14^1 + 1 = 15 \) because this sum equals 15 times the integer 1. Now suppose that 15 divides \( 14^{2k-1} + 1 \) where \( k \) is an integer greater than or equal to 1, so that by definition of “divides,” \( 14^{2(k-1)} + 1 = 15a \) for some integer \( a \). Consider \( 14^{2(k+1)-1} + 1 \). We have that
\[
14^{2(k+1)-1} + 1 = 14^{2k+1} + 1 \quad \text{by algebra}
= 14^2 \cdot 14^{2k-1} + 1 \quad \text{by algebra}
= 14^2(14^{2k-1} + 1 - 1) + 1 \quad \text{again by algebra}
= 14^2(14^{2k-1} + 1) - 14^2 + 1 \quad \text{by the Distributive Property}
= 14^2(14^{2k-1} + 1) - 195 \quad \text{again by the Distributive Property}
= 14^2 \cdot 15a - 195 \quad \text{by substitution from our assumption}
= 15(14^2a - 13) \quad \text{by algebra once more}
\]
In this last expression, \( 14^2a - 13 \) is an integer because it involves the difference, product, and WHOLE NUMBER POWER of integers and therefore is itself an integer. Thus, by definition, \( 15 \mid 14^{2(k+1)-1} + 1 \), and by PMI, it is true that \( 15 \mid (14^{2n-1} + 1) \) for all integers \( n \geq 1 \).

(c) \( 2 + 5n \leq 3^n \) for all integers \( n \geq ? \), where you determine the correct “base case”

The statement is true for all integers \( n \geq 3 \). When \( n = 3 \), we have the true statement \( 2 + 15 \leq 3^3 = 27 \). Suppose that \( 2 + 5k \leq 3^k \) where \( k \geq 3 \) is an integer. Consider \( 2 + 5(k+1) \), for which we see
\[
2 + 5(k+1) = 2 + 5k + 5 \quad \text{by algebra}
\leq 3^k + 5 \quad \text{by substitution from our assumption}
< 3^k + 3^k \quad \text{because 5 is less than any “legal” power of 3 in this problem}
= 2 \cdot 3^k \quad \text{by algebra}
< 3 \cdot 3^k \quad \text{by algebra but ONLY because } 3^k \text{ is POSITIVE!}
= 3^{k+1} \quad \text{by algebra one last time}
So $2 + 5(k + 1) \leq 3^{k+1}$ (we actually proved the stronger result that $2 + 5(k + 1) < 3^{k+1}$, but that certainly makes the $\leq$ claim true also, because $\leq$ represents an “or” statement), whence by PMI it is true that $2 + 5n \leq 3^n$ for all integers $n \geq 3$.

(To be rigorous, one really should formally prove the claim that 5 is less than any “legal” power of 3 in this problem – namely, that $5 < 3^k$ for all integers $k \geq 3$; notice that this subproof would require induction as well. However, the constant function 5 being compared to the elementary exponential function $3^k$ is basic enough precalculus knowledge that I will allow us to omit that proof. Remember, though, that subclaims comparing TWO NONCONSTANT functions – say $k + 5$ versus $3^k + k$ – are non-obvious and therefore entirely FAIR GAME for you at least to attempt the additional induction proof. See Problem #5 below.)

So as explained in class, our rule for whether you should or shouldn’t prove inductive subclaims will be that if both functions are constant (like 5), power functions (like $x$, $x^2$, $x^3$, etc.), or elementary exponential functions (like $2^k$, $3^k$, $4^k$, etc.), you need NOT prove the comparison you claim, but if at least one isn’t such a function (and that includes basic linear functions such as $x + 2$ and other binomials, you should try to prove it. (Style-wise, you could make such claims into lemmas and prove them before beginning the proof of the “bigger” result.)

(d) $2^{2n} + 5^n < 9^n$ for all integers $n \geq ?$, where you determine the correct “base case”

This claim is true for all integers $n \geq 2$. For the base case of $n = 2$, we see that $2^4 + 5^2 = 16 + 25 = 41 < 9^2 = 81$. Suppose that $2^{2k} + 5^k < 9^k$ where $k$ is an integer and $k \geq 2$, and consider $2^{2(k+1)} + 5^{k+1}$. We have

\[
2^{2(k+1)} + 5^{k+1} = 2^{2k+2} + 5^{k+1} \quad \text{by algebra}
\]

\[
= 2^2 \cdot 2^{2k} + 5 \cdot 5^k \quad \text{also by algebra}
\]

\[
= 2^2(2^{2k} + 5^k - 5^k) + 5 \cdot 5^k \quad \text{by algebra again}
\]

\[
= 2^2(2^{2k} + 5^k) - 2^2 \cdot 5^k + 5 \cdot 5^k \quad \text{by the Distributive Property}
\]

\[
= 2^2(2^{2k} + 5^k) + 5^k \quad \text{by algebra once more}
\]

\[
< 2^2 \cdot 9^k + 5^k \quad \text{by substitution from our inductive hypothesis}
\]

\[
< 2^2 \cdot 9^k + 9^k \quad \text{because $5^k < 9^k$ for all “legal” values of $k$ here}
\]

\[
= 5 \cdot 9^k \quad \text{by algebra}
\]

\[
< 9 \cdot 9^k \quad \text{by algebra because $9^k$ is positive}
\]

\[
= 9^{k+1} \quad \text{by algebra}
\]

(No need for a second inductive proof of the claim about $5^k$ vs. $9^k$ because they are both elementary exponentials)

So that $2^{2(k+1)} + 5^{k+1} < 9^{k+1}$, and by PMI, $2^{2n} + 5^n < 9^n$ for all integers $n \geq 2$.

5. [6 pts] Prove via mathematical induction: $(n+2)! < n^{2n}$ for all integers $n \geq ?$, where you determine the correct “base case.” This result likely has a hidden secondary inequality involved that you should prove via induction also.

This result is true for all integers $n \geq 3$. In the base case where $n = 3$, we have that $(3 + 2)! = 5! = 120 < 3^6 = 729$. Suppose that $(k+2)! < k^{2k}$ where $k$ is an integer with...
\( k \geq 3 \). Consider \( ((k+1) + 2)! \). We obtain

\[
((k+1) + 2)! = (k+3)! \quad \text{by algebra}
\]

\[
= (k+3) \cdot (k+2)! \quad \text{still by algebra}
\]

\[
< (k+3) \cdot k^{2k} \quad \text{by substituting from our assumption}
\]

\[
< k^2 \cdot k^{2k} \quad \text{see Lemma below}
\]

\[
= k^{2k+2} \quad \text{by algebra}
\]

\[
= k^{2(k+1)} \quad \text{by algebra}
\]

\[
< (k+1)^{2(k+1)} \quad \text{because } k < k+1 \text{ for ALL } k \text{ and both numbers are positive}
\]

Thus, \( ((k+1) + 2)! < (k+1)^{2(k+1)} \), so by PMI it's true that \( (n+2)! < n^{2n} \) for all integers \( n \geq 3 \).

**Lemma:** \( n + 3 < n^2 \) for all integers \( n \leq 3 \).

**Proof.** When \( n = 3 \), it is true that \( 6 < 3^2 \). Suppose that \( k + 3 < k^2 \) where \( k \) is an integer greater than or equal to 3. We have that

\[
(k + 1) + 3 = (k + 3) + 1 \quad \text{by the Commutative, Associative Properties (I’m tired of “algebra”)}
\]

\[
< k^2 + 1 \quad \text{due to substitution from our inductive hypothesis}
\]

\[
< k^2 + 2k + 1 \quad \text{by algebra because } 2k \text{ is POSITIVE}
\]

\[
= (k+1)^2 \quad \text{by algebra}
\]

We’ve shown that \( (k+1) + 3 < (k+1)^2 \), so by PMI, it’s true that \( n + 3 < n^2 \) for all integers \( n \leq 3 \).