1. (a) [6 pts] Without using the word “not,” state the negation of “For every point $P$, there exist points $Q$ and $R$ such that $P$, $Q$, and $R$ are noncollinear.”

There exists a point $P$ such that, for all points $Q$ and $R$, $P$, $Q$, and $R$ are collinear.

(Notice that this says something a little different from “There exists a point $P$ for all points $Q$ and $R$, such that $P$, $Q$, and $R$ are collinear.” The correct negation promises that a single point $P$ is collinear with any other points we ever encounter. The wrong negation only promises that every pair of points has its own “partner point” $P$; that is, different pairs of $Q$ and $R$ points might use different points $P$.)

(b) [6 pts] Without using the word “not,” state the contrapositive of “If $P$ and $Q$ are distinct points, then there exists a point $R$ for which $P$, $Q$, and $R$ are noncollinear.”

If for all points $R$, we have $P$, $Q$, and $R$ collinear, then $P$ and $Q$ are the same point/are equal.

(c) [6 pts] Rewrite the following statement using the phrase “is sufficient” in a mathematically meaningful way: “Two distinct lines are nonparallel only if there exists a unique point $P$ that lies on both.”

Being distinct and nonparallel is sufficient for two lines to have a unique point that lies on both.

(By the way, it is true that “Two distinct lines are nonparallel if and only if there exists a unique point $P$ that lies on both.” However, the statement in this problem wasn’t phrased that way.)

2. [18 pts - 6 each] Tell whether each definition below satisfies our criteria for a “good” definition. If not, name one that fails, and then correct it.

(a) Parallel lines are when they’re in the same plane but don’t intersect.

This one doesn’t have a good “class”; rephrase it as:
Parallel lines are lines in the same plane that don’t intersect.

(b) Noncollinear points are three or more distinct points that do not lie on the same line.

This one is fine: it’s a complete, reversible sentence that names the term being defined, it uses the class “points,” and it uniquely describes such points.

(c) A rational number is a real number of the form $\frac{a}{b}$ where $a$ and $b$ are integers with $b \neq 0$.

We know this one doesn’t uniquely describe rational numbers; it needs to be:
A rational number is a real number that can be written in the form $\frac{a}{b}$ where $a$ and $b$ are integers with $b \neq 0$.

(If we change the definition to “A rational number is a real number of the form $\frac{a}{b}$ in lowest terms, where $a$ and $b$ are integers with $b \neq 0$,” we still don’t get uniqueness of description: the number $3\pi/\pi$ is definitely a rational number, but it’s not in that particular form as is.)
3. (a) [8 pts] Clearly and completely describe how the undefined terms from Incidence Geometry are interpreted in the Klein disk model.

Points are ordinary points that lie inside a chosen circle. Lines are the intersections of ordinary lines with the interior of this circle. “Lie on” has the usual meaning.

(b) [8 pts] Name and state the parallel postulate that this model satisfies.

The Klein disk satisfies the Hyperbolic Parallel Postulate: Given any line \( \ell \) and any point \( P \) not on \( \ell \), there exist at least two distinct lines through \( P \) that are parallel to \( \ell \).

(Technically, this postulate does not promise more than two parallel lines, let alone infinitely many.)

4. [24 pts - 12 each] Determine whether each interpretation below fails or satisfies each of the axioms for Incidence Geometry, justifying your claims.

(a) Points are current students at SRU. Lines are classes offered at SRU this fall. A point lies on a line if the student is currently taking that class.

Axiom #1 fails: You can easily find two students who aren’t taking a class together (Ryan S. and Debbie A. are an example.)

Axiom #2 holds: Every class has at least two students in it, otherwise it wouldn’t be allowed to be offered. (If you said this fails because you know someone who’s taking an independent study course 1-on-1 with a prof, that’s okay too.)

Axiom #3 holds: You can easily find three different students who aren’t all in any class together. (Ryan S., Debbie A., and Will G. are an example.)

(Saying that they’re not in a particular class together isn’t quite the same as saying that they’re not in any classes together.)

(b) Points are real numbers. Lines are quadratic equations with real coefficients. A point lies on a line if that real number is a solution to the equation.

Axiom #1 fails: If you take any two distinct real numbers \( a \) and \( b \), they are the solutions to both \((x - a)(x - b) = 0\) and \(2(x - a)(x - b) = 0\), so they don’t determine a unique line.

(I had originally intended the equations in this problem to be monic: that means the leading coefficient is 1. With that restriction, Axiom #1 would hold, for then if you take any two distinct real numbers \( a \) and \( b \), they are the solutions to \((x - a)(x - b) = 0\), which is quadratic and will have real coefficients upon simplification, but it is the only such equation with a leading coefficient of +1.)

Axiom #2 fails: Some quadratic equations have no real solutions, such as \( x^2 + 1 = 0 \). Others have only one, such as \((x - 1)^2 = 0 \). Therefore, not all lines contain two points.

Axiom #3 holds: The numbers 4, 5, and 6 are noncollinear, since there is no quadratic equation that can have three different solutions.

5. [24 pts - 12 each] Carefully prove the following results from Incidence Geometry:

(a) If \( P \) is any point, then there exists at least one line \( \ell \) such that \( P \) does not lie on \( \ell \).

**Proof One.** Let \( P \) be a point, and let \( A \), \( B \), and \( C \) be the noncollinear points of Axiom #3. (Recall that \( A \), \( B \), and \( C \) are assumed to be distinct.)
**Case 1:** Suppose $P$ is one of these points; without loss of generality, let $P = A$. Then $P$ cannot lie on line $\overline{BC}$, which exists by Axiom #1, else $A$, $B$, and $C$ are collinear, a contradiction.

**Case 2:** Suppose $P$ is not any of $A$, $B$, or $C$. Consider **Axiom #1** lines $\overline{AB}$ and $\overline{AC}$, which are distinct else $A$, $B$, and $C$ would be collinear. $P$ cannot lie on both of these lines for then they’d intersect in two distinct points, $A$ and $P$, violating a previous theorem. Thus, $P$ does not lie on at least one of these lines.

**Proof Two.** Suppose not. Then there exists a point $P$ for which $P$ lies on every line $\ell$. Let $A$, $B$, and $C$ be the noncollinear points of Axiom #3. Because they are (assumed) distinct, the lines $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$ exist. Furthermore, these three lines are distinct, else $A$, $B$, and $C$ would be collinear. By assumption, $P$ must lie on all of these lines, but then $\overline{AB}$ and $\overline{AC}$, for example, are distinct nonparallel lines intersecting in at least two distinct points, $A$ and $P$. The contradicts a previous theorem.

(b) If $\ell$ is any line, then there exist lines $m$ and $n$ such that $\ell$, $m$, and $n$ are distinct and both $m$ and $n$ intersect $\ell$.

**Proof.** Let $\ell$ be a line. By Axiom #2, we know that it must contain distinct points $Q$ and $R$; by an earlier theorem, we know that there is a point $P$ that $\ell$ does not contain. Since $P$ is not on $\ell$, it is distinct from $Q$ and $R$, and therefore lines $m = \overline{PQ}$ and $n = \overline{PR}$ exist by Axiom #1. Certainly, these lines are different from $\ell$, since they contain point $P$ but $\ell$ does not. They are also distinct from each other, else $P$ is collinear with $Q$ and $R$, and lies on the unique line that they determine, but that can only be $\ell$. We therefore have three distinct lines, where $m$ intersects $\ell$ at $Q$ and $n$ does so at $R$ by their creation.